ON THE NONLINEAR THEORY OF HYDRODYNAMIC STABILITY

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PMM Vol.29, № 5, 1965, pp. 852-855

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(Received May 28, 1965)

Arnol'd [1] has suggested a method for investigating the stability of steady-state hydrodynamic flows that does not require the linearization of equations. His technique is based on the possibility of constructing a functional in the instantaneous states of the hydrodynamic fields which is conserved by virtue of the equations of motion and has a given steady-state flow as its extremal (stationary) point. If this extremum is a true minimum or true maximum, the flow is stable. The meaning of this theorem is geometrically clear. If we imagine the "level surfaces" of the functional in the functional space in the neighborhood of a point representing a given steady-state flow, then in the case of a maximum (minimum) they will be imbedded in one another by closed surfaces contracting to a point. If the steady-state flow is disturbed at some instant, the corresponding phase point will shift into the nearby level surface and will remain there throughout the intire subsequent time of motion by virtue of the conservation of the functional. A small initial deviation entails correspondingly small deviations throughout the entire subsequent time of motion.

In [1], where the matter under consideration was that of the two-dimensional flow of an incompressible ideal fluid, the existence of the required functional followed from two conservation laws: conservation of energy and conservation of vorticity. In the case of three-dimensional motion, the situation becomes somewhat more complicated. With a homogeneous and incompressible fluid, we no longer have a local invariant characteristic such as the curl of the velocity. The sole parameter which is conserved is the vorticity flux through any fluid area. In [2] Arnol'd constructs a generalization of the above method which includes the latter case but involves consideration of very unwieldy implicit expressions for surfaces in the functional space, these surfaces no longer being the level surfaces of certain functionals.

The purpose of the present paper is to show that there exists a class of flows for which such a difficulty does not arise, to wit — the adiabatic flows of an essentially nonhomogeneous fluid which, in addition, is assumed compressible.

It is possible to consider a nonhomogeneous and incompressible fluid. Dealing with the seemingly more complicated case of a nonhomogeneous fluid actually turns out to be more convenient due to the existence in this case of an invariant characteristic, the so-called "potential vorticity"

$$\Omega = \frac{\operatorname{grad} s \cdot \operatorname{rot} \mathbf{v}}{\rho} \tag{0.1}$$

where s is the entropy, ρ the density, and $\mathbf{v}$ the flow velocity (see, for example, [3]). If an incompressible fluid were being considered, it would be possible instead to write $\Omega = \mathbf{grad}\, \rho \cdot \mathbf{rot}\, \mathbf{v}$. Flows of this type are important, for example, in the meteorological study of zonal streams directed along the parallels.

1. Let us write the equations of motion. With a view toward meteorological application, we immediately introduce a coordinate system rotating uniformly with the angular velocity $\boldsymbol{\omega}$. The Euler equations can then be written as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\varphi - 2\boldsymbol{\omega}\times\mathbf{v}, \quad \frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0$$

$$\frac{ds}{dt} = 0, \qquad s = c_v \ln p\,\rho^{-\varkappa} \qquad (1.1)$$

where $\mathbf{v}$ is the velocity of relative motion, p the pressure, φ the potential energy of the external forces, e.g. of the weight field including the centrifugal forces of rotational transport motion, and s the entropy. By introducing the enthalpy w, we can rewrite the Euler equation as

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad} w + T\operatorname{grad} s - \operatorname{grad}\varphi - 2\boldsymbol{\omega}\times\mathbf{v}, \quad w = c_v T + \frac{p}{\rho} = c_p T \quad (1.2)$$

As already mentioned, there exists in addition to entropy yet another conserved quantity, the potential vorticity

$$\Omega = \frac{\operatorname{grad} s\cdot\boldsymbol{\zeta}}{\rho}; \qquad \frac{d\Omega}{dt} = 0 \qquad (\boldsymbol{\zeta} = \operatorname{rot}\mathbf{v} + 2\boldsymbol{\omega}) \qquad (1.3)$$

where ζ is the absolute vorticity.

We now investigate the stability of some steady-state flow $\mathbf{v}_0$, s_0, ρ_0, imposing on it two limitations. The first of these replaces the condition of monotony of the vorticity assumed in the two-dimensional case [1 and 2]. Specifically, we assume that the level surfaces of s_0 do not anywhere come in contact with the level surfaces of Ω_0,

$$\operatorname{grad} s_0 \times \operatorname{grad}\Omega_0 \neq 0 \qquad (1.4)$$

Each flow line is thereby unambiguously defined by a pair of values (s_0, Ω_0). The second limitation is as follows: we assume that the hard surface is one of the surfaces of constant entropy. Both of these limitations were first introduced in [4] in connection with the stability of atmospheric jet streams.

2. The steady-state flow we are considering obeys the Bernoulli theorem

$$\frac{\mathbf{v}_0^2}{2} + w_0 + \varphi = k(s_0, \Omega_0) \qquad (2.1)$$

where $k(s_0, \Omega_0)$ is some constant along the streamline. This formula has several corollaries. Let us take the gradient of both sides of this equation,

$$\mathbf{v}_0\times\operatorname{rot}\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0 + \operatorname{grad}(w_0+\varphi) = k_s\operatorname{grad} s_0 + k_\Omega \operatorname{grad}\Omega_0$$

or, recalling Euler equation (1.2),

$$\mathbf{v}_0\times(\operatorname{rot}\mathbf{v}_0 + 2\boldsymbol{\omega}) + T_0\operatorname{grad} s_0 = k_s\operatorname{grad} s_0 + k_\Omega\operatorname{grad}\Omega_0 \qquad (2.2)$$

Next, we multiply vectorially both sides of this equation by $\operatorname{grad} s_0$ and transform according to the double vector product formula

$$-\mathbf{v}_0(\operatorname{grad} s_0\cdot\boldsymbol{\zeta}_0) + \boldsymbol{\zeta}_0(\operatorname{grad} s_0\cdot\mathbf{v}_0) = k_\Omega\operatorname{grad}\Omega_0\times\operatorname{grad} s_0$$

The term $\operatorname{grad} s_0 \cdot \mathbf{v}_0$ is equal to zero because s_0 is constant along the streamline. The first term is $-\rho_0 \Omega_0 \mathbf{v}_0$. Thus,

$$\rho_0 \mathbf{v}_0 = \frac{k_\Omega}{\Omega_0} \operatorname{grad} s_0 \times \operatorname{grad} \Omega_0 \tag{2.3}$$

The meaning of this formula is evident. The velocity is directed along the tangent to the level surface of the conserved quantities s_0 and Ω_0. In addition, the area of the tube of flow is inversely proportional to $|\operatorname{grad} s_0 \times \operatorname{grad} \Omega_0|$. By the law of mass conservation, the proportionality factor between $\rho_0 \mathbf{v}_0$ and $\operatorname{grad} s_0 \times \operatorname{grad} \Omega_0$ must be constant on the streamline.

Now let us multiply both sides of (2.2) vectorially by $\operatorname{grad} \Omega_0$. In exactly the same way, transforming by the double vector product formula and applying Formula (2.3), we obtain

$$\frac{\operatorname{grad} \Omega_0 \cdot \zeta_0}{\rho_0} = (T_0 - k_s) \cdot \frac{\Omega_0}{k_\Omega} \tag{2.4}$$

The quantity in the left side is constructed similarly to the potential vorticity, except that it is Ω_0 instead of s_0 which is conserved.

3. Let us write out the functionals which are conserved by virtue of the laws of motion. The first of these is the energy integral (see [5], p.24),

$$\frac{dE}{dt} = 0, \qquad E = \iiint \rho \left(\frac{\mathbf{v}^2}{2} + c_v T + \varphi\right) dV$$

The second follows from the conservation of s and Ω for the individual particles,

$$\frac{dF}{dt} = 0, \qquad F = \iiint \rho \Phi(s, \Omega)\, dV$$

where Φ is an arbitrary function of two variables. We will show that this function can always be chosen such that the functional $I = E + F$ has a given steady-state flow as its stationary point, i.e. that the first variation of this functional becomes zero at that point. We vary this functional, considering $\delta \mathbf{v}$, δs, $\delta \rho$ as independent variations and expressing δT in terms of them through Formula

$$\delta T = T \delta s / c_\iota + p \delta \rho / \rho^2 c_v$$

We find that

$$\delta I = \iiint \Big\{ \rho \mathbf{v}\, \delta \mathbf{v} + \frac{1}{2} v^2 \delta \rho + w \delta \rho + \rho T \delta s + \varphi \delta \rho + \Phi \delta \rho + \\ + \rho \Big[\Phi_s \delta s + \Phi_\Omega \Big(\frac{\operatorname{grad} \delta s \cdot \zeta}{\rho} + \frac{\operatorname{grad} s \cdot \operatorname{rot} \delta \mathbf{v}}{\rho} - \frac{\Omega}{\rho} \delta \rho \Big) \Big] \Big\} dV$$

Transforming the terms containing $\operatorname{grad} \delta s$ and $\operatorname{rot} \delta \mathbf{v}$ by integrating by parts, we have

$$\iiint \Phi_\Omega \operatorname{grad} \delta s \cdot \zeta dV = - \iiint (\Phi_{\Omega\Omega} \operatorname{grad} \Omega + \Phi_{\Omega s} \operatorname{grad} s) \cdot \zeta \delta s dV = \\ = - \iiint \rho \Big[\Phi_{\Omega\Omega} \frac{\operatorname{grad} \Omega \cdot \zeta}{\rho} + \Phi_{\Omega s} \cdot \Omega \Big] \delta s dV$$

$$\iiint \Phi_\Omega \operatorname{grad} s \cdot \operatorname{rot} \delta \mathbf{v}\, dV = - \iiint \Phi_{\Omega\Omega} \operatorname{grad} s \times \operatorname{grad} \Omega \cdot \delta \mathbf{v}\, dV$$

This gives us

$$\delta I = \delta I_1 + \delta I_2 + \delta I_3$$

where

$$\delta I_1 = \iiint (\rho \mathbf{v} - \Phi_{\Omega\Omega} \operatorname{grad} s \times \operatorname{grad} \Omega) \delta \mathbf{v} \, dV$$

$$\delta I_2 = \iiint \rho \left(T + \Phi_s - \Phi_{\Omega\Omega} \cdot \frac{\operatorname{grad} \Omega \cdot \boldsymbol{\zeta}}{\rho} - \Phi_{\Omega s} \cdot \Omega \right) \delta s dV$$

$$\delta I_3 = \iiint \left(\frac{v^2}{2} + w + \varphi + \Phi - \Omega \cdot \Phi_\Omega \right) \delta \rho dV$$

In order for the variation to equal zero it is necessary that the coefficients of the independent variations go to zero when $\mathbf{v}$, s and ρ are replaced by quantities referring to our steady-state flow. Taking into account (2.1),(2.3) and (2.4), we obtain from this the three equations

$$k_\Omega / \Omega - \Phi_{\Omega\Omega} = 0, \quad \Phi_s + k_s - \Phi_{\Omega s} \cdot \Omega = 0, \quad k + \Phi - \Phi_\Omega \cdot \Omega = 0$$

of which only one, the third, is independent, while the first two are its direct corollaries. Specifically, the first may be obtained by differentiating the third with respect to Ω , and the second by differentiating it with respect to s . Thus, if as our Φ we take a function satisfying Equation

$$k + \Phi - \Phi_\Omega \cdot \Omega = 0 \tag{3.1}$$

then the functional I constructed for a given flow has a stationary value.

Without giving its derivation, let us merely set down the expression for the second variation,

$$\delta^2 I = \iiint \Big[\rho \delta v^2 + 2\mathbf{v} \, \delta \mathbf{v} \, \delta \rho + \frac{p}{\rho^2} \delta \rho^2 + \frac{\rho k_\Omega}{\Omega} \left(\frac{T - k_s}{k_\Omega} \delta s - \delta \Omega \right)^2 +$$

$$+ \frac{p}{c_v R} \left(\delta s + \frac{R}{\rho} \delta \rho \right)^2 - \frac{\operatorname{grad} T \cdot \boldsymbol{\zeta}}{\Omega} \delta s^2 - \frac{2}{\Omega} \mathbf{v} \times \boldsymbol{\zeta} \cdot \operatorname{rot} \delta \mathbf{v} \, \delta s \Big] dV \tag{3.2}$$

If this were to turn out positive for some flow, that flow would be stable. The last term of the second variation apparently excludes such a possibility, as is the case in the paper of Arnol'd [2]. It is conceivable that the resulting expression might throw some light on the mechnism involved in loss of stability.

The author is grateful to V.I.Arnol'd, whose work suggested the subject of the present paper.

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Translated by A.Y.